INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2002; **38**:255–282 (DOI: 10.1002/fld.217)

An analysis and comparison of the time accuracy of fractional-step methods for the Navier–Stokes equations on staggered grids

S. Armfield[1,*,†] and R. Street[2]

[1]*Department of Mechanical and Mechatronic Engineering, Sydney University, Sydney, Australia 2006*
[2]*Environmental Fluid Mechanics Laboratory, Stanford University, Stanford, CA 94305-4020, U.S.A.*

SUMMARY

Fractional-step methods solve the unsteady Navier–Stokes equations in a segregated manner, and can be implemented with only a single solution of the momentum/pressure equations being obtained at each time step, or with the momentum/pressure system being iterated until a convergence criterion is attained. The time accuracy of such methods can be determined by the accuracy of the momentum/pressure coupling, irrespective of the accuracy to which the momentum equations are solved. It is shown that the time accuracy of the basic projection method is first-order as a result of the momentum/pressure coupling, but that by modifying the coupling directly, or by modifying the intermediate velocity boundary conditions, it is possible to recover second-order behaviour. It is also shown that pressure correction methods, implemented in non-iterative or iterative form and without special boundary conditions, are second-order in time, and that a form of the non-iterative pressure correction method is the most efficient for the problems considered. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: Navier–Stokes; staggered; projection; fractional step

1. INTRODUCTION

Time integration of the Navier–Stokes equations is often carried out by means of the fractional-step procedure, first suggested by Harlow and Welch [1] and Chorin [2]. With Chorin's method at each time step an incomplete form of the momentum equations is integrated to yield an approximate velocity field, which will in general not be divergence free, then a correction is applied to that velocity field to produce a divergence-free velocity field. The correction to the velocity field is an orthogonal projection in the sense that it projects the initial velocity field onto the divergence-free field without changing the vorticity. This step is called the projection step, and schemes that use this approach are often called projection methods. The original Chorin method was modified for use with finite volumes defined on a staggered grid by Kim

* Correspondence to: S. Armfield, Department of Mechanical & Mechatronic Engineering, Sydney University, Sydney, NSW 2006, Australia.
† E-mail: armfield@mech.eng.usyd.edu.au

Received 15 November 1999
Revised 8 November 2000

and Moin [3], and has since been used by many researchers for the simulation of unsteady flows [4]. The Harlow and Welch scheme led to the development of the SIMPLE scheme of Patankar and Spalding [5]. This iterative scheme, which was initially applied to the fully implicit steady-state Navier–Stokes equations, has since been applied very widely for both steady and unsteady flow.

In this paper we will investigate non-iterative semi-implicit schemes similar to that of Kim and Moin. The Kim and Moin finite volume method drops the pressure gradient from the momentum equations, which are integrated to obtain an intermediate velocity field using an approximate time split method. A Poisson equation is then solved for a form of the pressure, which is used to correct the velocities and generate a divergence-free field. The integration then proceeds to the next time step. Considerable effort has been expended on determining appropriate boundary conditions for the intermediate velocity field and the pressure Poisson equation. The use of the physical velocity boundary conditions and zero normal pressure gradient, as is normal with equivalent iterative schemes, can result in the scheme being first-order in time, irrespective of the time accuracy with which the momentum equations have been solved. The original Kim and Moin scheme used a modified second-order boundary condition. Karniadakis *et al.* [6] showed that a boundary condition could be obtained for the Poisson equation that resulted in second-order in time accuracy for a simpler projection scheme, using spectral elements for the spatial discretization.

Perot [7] examined similar projection methods using an LU factorization scheme which required no boundary conditions for the intermediate velocity and pressure, and showed that the scheme was first-order. Perot attributed the first-order behaviour to a commutation error, and suggested that this was an intrinsic feature of the projection method that could not be generally remedied by modifying the boundary conditions. Perot proposed a modification to the LU factorization form of the projection method that produced second-order time accuracy.

An alternative method of obtaining second-order in time behaviour for projection methods has been suggested by a number of authors. These methods retain the pressure gradient in the momentum equations, using the best available estimate for the pressure. A pressure correction is then obtained simultaneously with the enforcement of the divergence-free condition for the velocity field. Both Van Kan [8] and Bell and Colella [9] have suggested and analysed pressure correction methods of this type. The schemes of Van Kan and Bell and Colella were non-iterative in the sense that the momentum and pressure correction equation are only solved once at each time step. Van Kan presented results for the Navier–Stokes equations showing that the pressure correction scheme was second-order in time while Bell and Colella presented no time accuracy results. Gresho [10] carried out a detailed analysis of projection methods in a finite element context, labelling the basic method in which the pressure gradient is dropped from the momentum equations P1 and the pressure correction method P2. It was demonstrated analytically that P1 with physical boundary conditions for the intermediate velocity was first-order in time whereas P2 was second-order in time. Guermond and Quartapelle [11] also investigated P1 and P2 methods, focussing primarily on the effect of different spatial discretizations, but again demonstrated the first-order behaviour of the P1 method and the second-order behaviour of the P2 method. The pressure correction approach can also be applied iteratively, whereby the momentum and pressure correction equation system is solved repeatedly at each time step, ensuring that any error associated with the single iteration is minimized. This approach was suggested by Tau [12], although it appears that the results presented in that paper were actually obtained using the non-iterative method. Tau presented no time accuracy results. Dukowicz and Dvinsky [13] suggested an approximate factorization method that was similar to the pressure correction methods cited above, and demonstrated second-order time accuracy. Issa [14] suggested a fully implicit limited iteration pressure correction method in the context of an unsteady scheme whereby two iterations of the momentum, pressure correction equation system are carried out at each time step.

It is apparent that a range of projection/pressure correction methods have been developed and are being used for the simulation of unsteady incompressible flow, with a range of taxonomies being used to categorize the schemes. In this paper we will investigate iterative and non-iterative projection/pressure correction methods using a semi-implicit discretization in which the advective terms are treated explicitly and the diffusive terms are treated implicitly, following the Kim and Moin approach. The schemes will be labelled following the Gresho taxonomy as follows:

- P1 projection: the basic projection method with the pressure gradient dropped from the momentum equations and a single iteration of the momentum/pressure equation system carried out at each time step.
- P2 pressure correction: the basic pressure correction method with a single iteration of the momentum/pressure correction equation system carried out at each time step.
- Iterative: the pressure correction method with repeated iterations of the momentum/pressure correction equation system carried out at each time step.

There has been little direct comparison of the accuracy and efficiency of the P1, P2 and iterative methods. While many of the authors cited earlier demonstrated the order of accuracy of the schemes considered, few directly compared the accuracy of the methods and none considered the comparative efficiency of the schemes. Recently Armfield and Street [15] compared the accuracy and efficiency of the P1 projection and P2 pressure correction methods with an iterative method and demonstrated that the P2 pressure correction method is second-order accurate in time and is the most efficient of the methods considered. In the present paper we considerably extend the previous work and investigate the time accuracy and efficiency of the P1 projection method, a modified second-order P1 projection method based on Perot's scheme, the Kim and Moin P1 method, a P2 pressure correction method, and an iterative method. It is shown that the first-order projection error is a commutation error, as Perot noted, but that it occurs only at the boundary and can be modified by a change in boundary conditions for the intermediate velocity, as suggested by Kim and Moin. The Kim and Moin boundary conditions replace the first-order error with a second-order error which is equivalent to the projection error of the P2 pressure correction scheme in which standard boundary conditions are used for the intermediate velocity field. It is also shown that by implementing Kim and Moin-type boundary conditions with the P2 pressure correction method, but using the pressure correction rather than the pressure, the second-order projection error is replaced by a third-order error, yielding the most accurate and efficient scheme of those considered.

2. METHOD

The governing equations are the Navier–Stokes equations is unsteady incompressible non-dimensional form,

$$u_t + (u \cdot \nabla)u = -\nabla P + \frac{1}{Re}\nabla^2 u \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

where u is the velocity, P the pressure and Re the Reynolds number.

The continuous equations are discretized using Adams–Bashforth for the advective terms and Crank–Nicolson for the diffusive terms, giving the system,

$$\frac{v^{n+1} - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -Gp^{n+1} + \frac{1}{2Re}L(v^{n+1} + v^n) \tag{3}$$

$$v^{n+1} = 0 \tag{4}$$

where (v, p) are the discrete velocity and pressure respectively, H is the discrete advection operator, G the discrete gradient, L the discrete Laplace operator and D the discrete divergence. This is a second-order in time discretization, using an explicit scheme for the advection terms and an implicit scheme for the diffusion terms [3]. Fractional-step methods integrate Equations (3) and (4) in a segregated manner, that is the momentum equations are first solved for the velocity, and some form of Poisson equation is then solved for the pressure. The Poisson equation is constructed from the momentum equation and the continuity equation and, as well as providing the pressure, acts to enforce continuity.

The schemes considered in this paper first obtain an estimate for the velocity at the $n+1$ time level, by solving an approximation of Equation (3), and then solving a Poisson equation that is constructed by applying a correction to the estimated velocity field by requiring that it satisfies continuity. Of the methods to be considered the iterative pressure correction method is the most general in the sense that it provides an exact solution to the discrete Equations (3) and (4), at least to the degree of accuracy of the convergence criterion specified. The non-iterative P2 pressure correction method is functionally identical to the iterative pressure correction method, but with the momentum and Poisson equations being solved only once at each time step. The non-iterative P2 pressure correction method will not in general provide an exact solution to the discrete equations, regardless of the accuracy to which the individual equations are solved.

The P1 projection method is very similar to the P2 pressure correction method, but with the pressure term neglected from the momentum equations. The Poisson equation, which in the iterative and P2 pressure correction methods yields a pressure correction, provides a form of the pressure in the P1 projection method. The three methods are described next.

2.1. Iterative method

In this method Equation (3) is solved, using the best current value for p^{n+1}, to obtain v^*, an approximation to v^{n+1}, that is

$$\frac{v^* - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -Gp^{n+1} + \frac{1}{2Re}L(v^* + v^n) \tag{5}$$

This approximate velocity will not initially satisfy continuity. A correction is then applied of the form,

$$v^{n+1} = v^* - \Delta t G\pi \tag{6}$$

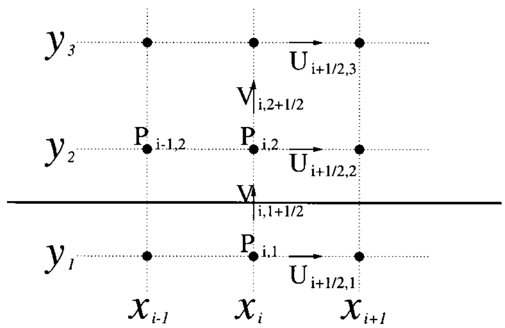

Figure 1. Staggered grid, solid circles indicate pressure storage locations, horizontal arrows indicate horizontal (U) velocity locations and vertical arrows indicate vertical (V) velocity locations. The solid line is the domain boundary with y_1 the exterior point and $y_2, y_3, \ldots$ interior points.

where π is a pressure correction, such that the resulting v^{n+1} does satisfy continuity. An equation for π is constructed by substituting Equation (6) into the continuity Equation (4), to give,

$$L\pi = Dv^*/\Delta t$$

Once π is obtained, the velocity is corrected and the pressure is updated using the pressure correction as,

$$p^{n+1} = p^{n+1} + \pi \tag{7}$$

Equation (5) is then solved again using the updated pressure to obtain a new estimate of the velocity at the $n+1$ time level, and that velocity again corrected to enforce continuity and provide a pressure correction. This process is repeated until the integral over the domain of the absolute divergence after the solution of Equation (5) is less than a predefined value. The solution is then said to be converged and the integration continues to the next time step. For the first iteration at each time step p^{n+1} is set equal to p^n.

At the completion of the time step the solution will satisfy both Equations (3) and (4), to within the required convergence condition, and is therefore expected to be second-order accurate in time, that is to have the same order of accuracy as that of the discretization of the momentum equations. In practice, the scheme is most efficient if it is not required that the pressure correction step results in a velocity field that satisfies the divergence-free condition at each iteration, rather it is only required that the divergence error be reduced, and over a number of momentum/pressure correction iterations the velocity will approach and satisfy the divergence-free condition.

2.1.1. Boundary conditions. In the next section results are presented for flows with three types of boundary conditions:

- Fixed velocity boundaries: both the U and V velocity components are fixed at a solid wall or an inlet boundary. In this case the normal component of velocity, which has a node on the boundary as shown in Figure 1, is set to the required value at that boundary, while the tangential component, which does not have a node on the boundary, has the

average of the values at the immediate interior and exterior nodes, set to the required value. Because the normal component of velocity is known at the boundary no correction is required to the * field there, and therefore the normal gradient of π is set to zero at the boundary.
- Zero tangential shear boundaries: the normal component of velocity is set to zero and the normal derivative of π is then also zero. The tangential velocity at the exterior node is set equal to that at the immediate interior node.
- Outlet boundary: the flow is assumed fully developed and the normal derivative of both velocity components is set to zero. From Equation (6) the normal-second derivative of π is set to zero.

The boundary conditions for the * velocity field are set to be the same as the physical boundary conditions, given above. No explicit boundary conditions are required for the pressure.

2.2. P1 Projection methods

In these methods Equation (3) without the pressure gradient term is solved to obtain v^*, that is

$$\frac{v^* - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = \frac{1}{2Re}L(v^* + v^n) \tag{8}$$

Again this approximate velocity field will not in general satisfy continuity and a correction is applied of the form,

$$v^{n+1} = v^* - \Delta t G\phi \tag{9}$$

such that the resulting v^{n+1} does satisfy continuity. An equation for ϕ is constructed by substituting Equation (9) into Equation (4), to give,

$$L\phi = Dv^*/\Delta t \tag{10}$$

Once ϕ is obtained, the velocity is corrected and the integration continues to the next time step. In the P1 projection method it is necessary to solve the Poisson equation for ϕ very accurately to ensure that the velocity remains divergence free. At each time step ϕ is initialized using the previous time step value. This significantly reduces the run time, by up to a factor of two, when compared to the run time for ϕ initialized to zero at each time step.

The basic P1 projection method uses the same boundary conditions for the * velocity field and ϕ as those given above in the iterative method for the * velocity field and π. The P1 projection method is thus a one-step method, and is therefore likely to require less computer time per time step than the iterative method. However, it is not possible to obtain a p^{n+1} such that the final (v^{n+1}, p^{n+1}) satisfies Equations (3) and (4) to within the convergence with which each of the equations has been solved, as shown below.

2.2.1. Accuracy. The velocity field v^{n+1}, obtained using the projection method, will be divergence free and satisfy the following approximation to the discrete momentum equations,

$$\frac{v^{n+1} - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -G\phi + \frac{\Delta t}{2Re}LG\phi + \frac{1}{2Re}L(v^{n+1} + v^n) \tag{11}$$

which is obtained by substituting $v^* = v^{n+1} + \Delta t G\phi$ into Equation (8). For this to equal Equation (3) above requires that

$$Gp^{n+1} = G\phi - \frac{\Delta t}{2Re} LG\phi \tag{12}$$

This will only be true if the right hand side can be expressed as the gradient of a scalar, which requires that the operators L and G commute. In continuous form this is the case, and it is also true in discrete form in the domain interior. However, it is not necessarily the case for the tangential momentum equation at the node immediately interior to the boundary, as will now be demonstrated.

We consider the U momentum equation at the $(x_{i+1/2}, y_2)$ location as shown in Figure 1. This is the location of the immediate interior tangential momentum equation, located $h/2$ vertically from the boundary and between the $P_{i,2}$ and $P_{i+1,2}$ pressure nodes. Nodal ϕ values are located at pressure nodes. Defining $y = 0$ to be a zero velocity boundary then gives the following physical boundary conditions for U, V and ϕ

$$V_{i,1+1/2} = 0, \quad U_{i+1/2,1} = -U_{i+1/2,2}, \quad \phi_{i,1} = \phi_{i,2}$$

Using the same boundary conditions for the * velocity field as the physical velocity, given above, the boundary condition term $U^*_{i+1/2,1} = -U^{n+1}_{i+1/2,2} - \Delta t \Delta\phi_{i+1/2,2}/h$. Writing the expression for the y component of

$$\frac{\Delta t}{2Re} LG\phi$$

explicitly for the U-momentum equation at the $(x_{i+1/2}, y_2)$ node gives,

$$\frac{\Delta t}{2Reh^3}[\Delta\phi_{i+1/2,3} - 2\Delta\phi_{i+1/2,2} - \Delta\phi_{i+1/2,2}] \tag{13}$$

where $\Delta\phi_{i+1/2,j} = (\phi_{i+1,j} - \phi_{i,j})$. Using the boundary condition $\phi_{i,1} = \phi_{i,2}$ this expression may be written as,

$$\frac{\Delta t}{2Reh^3}[\Delta\phi_{i+1/2,3} - 2\Delta\phi_{i+1/2,2} + \Delta\phi_{i+1/2,1}] - \frac{\Delta t}{Reh^3}\Delta\phi_{i+1/2,1}$$

The first term in this expression will now commute to give $GL\phi$, while the second-term is the error. The term

$$\frac{\Delta\phi_{i+1/2,1}}{h}$$

is the finite difference form of ϕ_x at $(x_{i+1/2}, y_1)$, and thus the error is of the form,

$$\frac{\Delta t}{Reh^2}\phi_x$$

which occurs only at the $j = 2$ node. Because this error occurs only at the boundary interior node with a divisor of $1/h^2$ it is consistent and leads to an error in the interior of the form,

$$\Delta t \phi_x$$

This projection error is of order Δt indicating that even though a second-order in time differencing has been used for the momentum equations, the basic projection method is expected to be only first-order in time as a result of this additional error.

This error is entirely dependent on the form of the boundary conditions. If, for instance, a zero tangential shear boundary condition is specified then the boundary conditions are

$$V_{i,1+1/2} = 0, \quad U_{i+1/2,1} = U_{i+1/2,2}, \quad \phi_{i,1} = \phi_{i,2}$$

and Equation (13) becomes,

$$\frac{\Delta t}{2Reh^3}[\Delta\phi_{i+1/2,3} - 2\Delta\phi_{i+1/2,2} + \Delta\phi_{i+1/2,1}]$$

which will commute to give GL, and no error is produced. Perot [7] also observed that for periodic boundary conditions no error occurs.

The error obtained above is a result of the implicit discretization of the diffusion terms, and could be eliminated by utilizing an explicit discretization for these terms. However, a severe restriction on the size of the time step is then necessary to ensure stability, and for this reason an implicit discretization of the type used here is commonly employed. Results in the next section are included for both the semi-implicit scheme given above and for the fully explicit scheme in which Adams–Bashforth is used for both the advection and diffusion terms.

The accuracy analysis above shows that the most accurate form of the pressure is obtained from Equation (12) with L and G commuted, that is

$$p^{n+1} = \phi - \frac{\Delta t}{2Re}L\phi \tag{14}$$

However, a simpler form of the pressure is often used, that is,

$$p^{n+1} = \phi \tag{15}$$

2.2.2. Kim and Moin method. Kim and Moin [3] derived a boundary condition for the * velocity field for fixed velocity boundaries that removes the error obtained above that occurs with the basic P1 projection method. Assuming zero velocity at the boundary their boundary condition for the * tangential velocity at the $x_{i+1/2}$ location is,

$$U^*_{i+1/2,1} = -U^*_{i+1/2,2} + \frac{2\Delta t}{h}\Delta\phi^n_{i+1/2,1}$$

Similar conditions are applied for fixed tangential velocity boundary conditions on all other boundaries. Normal velocity boundary conditions remain the same as in the basic P1 projection method, that is the physical boundary conditions are applied. Boundary conditions for ϕ, which depend on the normal velocity boundary conditions, therefore also remain as for the basic P1 projection method. The superscript n has been introduced to indicate the time location of ϕ. This is important as the single-step nature of the scheme means that the n level ϕ must be used in the tangential velocity boundary condition, whereas the $n+1$ level ϕ is used for the velocity correction.

Substituting the Kim and Moin boundary conditions for the tangential velocity into the tangential momentum equation at the $(x_{i+1/2}, y_2)$ location, as above, gives an error of the form

$$\frac{\Delta t}{2Reh^3}[\Delta\phi_{i+1/2,3} - 2\Delta\phi_{i+1/2,2} + \Delta\phi_{i+1/2,1}]^{n+1} - \frac{\Delta t}{Reh^3}\Delta(\phi^{n+1}_{i+1/2,1} - \phi^{n}_{i+1/2,1})$$

The first term in this expression will commute to give $GL\phi$ as required, and is thus part of the pressure. The second term is similar to the error term in the basic P1 projection method, but now depends on $(\phi^{n+1} - \phi^n)$, which is the finite difference form of $\Delta t\phi_t$ and is therefore of order Δt. The overall error is therefore of the form

$$\Delta t^2\phi_{tx} \tag{16}$$

which is second-order in time.

The boundary condition given above has been derived specifically for the fixed tangential velocity case and must be used with care when applied with other boundary conditions. For instance if the following boundary condition,

$$U^*_{i+1/2,1} = U^*_{i+1/2,2} + \frac{\Delta t}{h}\Delta\phi^n_{i+1/2,1} \tag{17}$$

is used with a zero shear boundary, the substitution into the tangential momentum equation gives,

$$\frac{\Delta t}{2Reh^3}[\Delta\phi^{n+1}_{i+1/2,3} - 2\Delta\phi^{n+1}_{i+1/2,j} + \Delta\phi_{i+1/2,1}]^{n+1} - \frac{\Delta t}{Reh^3}\Delta\phi^n_{i+1/2,1}$$

leading to a Δt error.

As noted above for the zero tangential shear the physical velocity boundary conditions should be applied. Using the Kim and Moin approach it is therefore necessary to determine appropriate tangential * velocity boundary conditions for each case of physical boundary conditions.

For the Kim and Moin method, as with the basic P1 projection method, the most accurate form of the pressure is obtained as,

$$p^{n+1} = \phi - \frac{\Delta t}{2Re}L\phi \tag{18}$$

while the simpler form of the pressure, which is often used, is

$$p^{n+1} = \phi \tag{19}$$

2.2.3. Perot. Perot [6] suggested a different approach whereby a modification to an *LU* factorization scheme is used to remove the first-order commutation error, adding an additional term to the velocity correction given in Equation (9), obtaining,

$$v^{n+1} = v^* - \Delta tG\phi - \frac{\Delta t^2}{2Re}LG\phi \tag{20}$$

This replaces the first-order error with an equivalent but second-order error. To implement this approach with the fractional step method requires that separate boundary conditions for the ϕ gradient be specified for each of the velocity components. For the correction applied to the U component the ϕ boundary conditions are specified as $\phi_x = 0$ on all boundaries. Conversely for the V component $\phi_y = 0$ on all boundaries.

Taking the divergence of this modified velocity correction to obtain an equation for ϕ leads to a higher order equation with a large computational stencil. Rather than solving this equation, the standard Poisson equation for ϕ, Equation (10), is treated as an equation for a ϕ correction. This equation is solved and the resulting ϕ is then used to correct the velocity as given in Equation (12), ϕ is set to zero and then the divergence of the corrected velocity used as the source for another solution of the ϕ equation, and the process repeated until the divergence satisfies the convergence condition. The boundary conditions for the ϕ Poisson equation are the same as those for the basic P1 projection method; the special conditions given above are only applied when the velocity is corrected. This iterative approach to the solution of the modified ϕ equation converges rapidly and leads to a much simpler computational scheme.

2.3. P2 pressure correction method

The pressure correction method is identical to the iterative method, but with only a single iteration carried out at each time step. The discrete momentum equation is solved to obtain v^*, that is

$$\frac{v^* - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -Gp^n + \frac{1}{2Re}L(v^* + v^n)$$

using the nth time-level pressure, as with the first iteration of the iterative method. The v^* field is then corrected to satisfy continuity and the pressure corrected in exactly the same way as in the iterative method. As the P2 pressure correction equation is only solved once each time step, it is necessary in the P2 pressure correction method to obtain an accurate solution of the Poisson equation, in the same way as the P1 projection method requires an accurate solution for ϕ.

2.3.1. Accuracy. The velocity field v^{n+1}, obtained using the pressure correction method, will satisfy the following approximate form of the momentum equations.

$$\frac{v^{n+1} - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -G(p^n + \pi) + \frac{\Delta t}{2Re}LG\pi + \frac{1}{2Re}L(v^{n+1} + v^n) \tag{21}$$

For this to equal Equation (3) requires that,

$$p^{n+1} = G(p^n + \pi) - \frac{\Delta t}{2Re}LG\pi$$

which again requires that L and G commute, and the $n+1$ pressure then becomes,

$$p^{n+1} = p^n + \pi - \frac{\Delta t}{2Re}L\pi$$

However, for fixed tangential velocity boundary conditions substitution of U^* into the immediate interior node momentum equation shows that the discrete L and G will not commute, exactly as for the basic P1 projection method, and the error is of the form,

$$\frac{\Delta t}{Re} LG\pi$$

This error is similar to that obtained with the P1 projection method, but is now dependent on the pressure correction π rather than the pseudo-pressure, ϕ. As π is an approximation of $\Delta t p_t$, the overall error will be of the form,

$$\Delta t^2 p_{tx} \tag{22}$$

and is second-order in time.

This error is very similar to that obtained for the Kim and Moin boundary condition and thus the two schemes are expected to perform similarly. A form of the Kim and Moin boundary condition can be used with the pressure correction method, whereby for a zero velocity boundary the tangential velocity is set to,

$$U^*_{i+1/2,1} = - U^*_{i+1/2,2} + \frac{2\Delta t}{h} \Delta\pi^n_{i+1/2,1} \tag{23}$$

in which case the error will be of the form,

$$\Delta t(\pi_x^{n+1} - \pi_x^n)$$

and is expected to be third-order in time. This is similar to the optimal P2 method suggested in Reference [10].

The most accurate pressure obtained with the pressure correction method is,

$$p^{n+1} = p^n + \pi - \frac{\Delta t}{2Re} L\pi \tag{24}$$

while the simpler form uses

$$p^{n+1} = p^n + \pi \tag{25}$$

2.4. Discretization

The above schemes are defined on the standard *MAC* staggered grid using finite volumes, with standard second-order central differences used for the viscous terms, the pressure gradient and divergence terms. The QUICK third-order upwind scheme is used for the advective terms [16]. The momentum equations are inverted using an ADI scheme in which terms are shifted to the right hand side of the system to enable a series of tridiagonal matrices to be inverted in each direction. The terms shifted to the right hand side contain the latest available estimate for the unknown, allowing the domain to be repeatedly swept until an accurate solution is obtained. For all the methods tested four sweeps of the ADI solver were used, where a single sweep consists of solving the series of tridiagonal systems associated with each coordinate direction once. Four sweeps of the solver gave solutions with residuals of less than 1×10^{-8} for all cases. Reducing the number of sweeps to one reduced the run time by a maximum of 10 per cent, for the iterative solver, while also marginally reducing the accuracy of the final solution. A preconditioned restarted GMRES method is used to solve the Poisson ϕ and pressure correction equations for all the methods. Other solvers, such as preconditioned conjugate gradient, incomplete LU, ADI and Jacobi have also been tested and found not to affect the overall accuracy or relative performance of the methods. Of the solvers tested GMRES was found to be the most efficient. The number of sweeps of the GMRES solver used varied with each of the methods tested and with the time-step and convergence criterion prescribed. For the non-iterative schemes for the smallest convergence criterion up to a hundred sweeps were required while for the largest convergence criterion as few as five were sufficient. For the iterative scheme the Poisson solver was limited to five sweeps as described in the next section. In all cases tested the majority of the computational time was used solving the Poisson ϕ and pressure correction equations, varying from slightly more than 50 per cent for the iterative solver with the largest time-step and convergence criterion to 95 per cent for the duct flow with the smallest time step and convergence criterion.

3. RESULTS

Results have been obtained for natural convection cavity flow, driven cavity flow and symmetric duct flow. Natural convection and driven cavity flows are standard benchmarks that have been used extensively for testing the convergence and accuracy of Navier–Stokes solvers. Symmetric duct flow has been included as an example of a problem with free tangential shear on one boundary. All results presented were obtained using double precision on a DEC 3000-700.

3.1. Natural convection results

Initially the fluid in the square cavity is stationary and isothermal at temperature $T=0$. At time $t=0$ the left and right walls are instantaneously heated and cooled to $\Delta T/2$ and $-\Delta T/2$ respectively, with the top and bottom boundaries adiabatic. All boundaries are no-slip. The control parameters for this flow are the Rayleigh number Ra and the Prandtl number Pr. The Rayleigh number $Ra=g\alpha\Delta TH^3/\nu\kappa$, with g gravity, α the coefficient of thermal expansion, H the height of the cavity, ν the kinematic viscosity and the diffisivity $\kappa=\nu/Pr$. The results presented were obtained with $Ra=6\times10^5$ and $Pr=7.5$.

The two-dimensional equations are used with x the horizontal co-ordinate, U the corresponding horizontal velocity component, y the vertical coordinate and V the corresponding vertical velocity component. The natural convection flow requires the inclusion and solution of a temperature equation, in addition to the Navier–Stokes equations. The temperature equation is solved using Adams–Bashforth and Crank–Nicolson schemes in exactly the same manner as the momentum equations, and for brevity is not presented. Further details of the natural convection flow may be found in Patterson and Armfield [17] and Armfield and Patterson [18], and for brevity will not be presented here.

A 50×50 uniform mesh has been used. The 50×50 solution was compared to that obtained on a 200×200 mesh and the variation was found to be less than 1 per cent. The 50×50 mesh is therefore considered to provide a sufficiently accurate resolution for this flow. To test the behaviour of the methods the flow was integrated from $t=0$ to $t=2$ for time steps in the range $\Delta t=0.003125$ to 0.1, and the 'error' expressed as the L_2 norm of the difference between a test solution obtained at a given Δt and a benchmark solution obtained with a time step of $\Delta t = 7.8125 \times 10^{-4}$, also integrated from $t = 0$ to $t = 2$. Times have been non-dimensionalized using the boundary layer start-up time for the natural convection cavity. Total time to steady state for the cavity is orders of magnitude greater than the boundary layer start-up time. The maximum time step selected, $\Delta t = 0.1$, was chosen to be near to the empirically obtained stability limit of $\Delta t = 0.2$ for the P1 projection, P2 pressure correction and iterative methods. The Perot and explicit methods are less stable and were restricted to a maximum time step of $\Delta t = 0.025$.

For each of the methods and time steps results have been obtained with convergence criterion ranging from 1.0×10^{-4} to 1.0×10^{-9} in order-of-magnitude steps. The solution was considered converged at each time step when the integral over the domain of the absolute residual of the continuity equation was less than the convergence criterion. In this way it was possible to determine which was the appropriate convergence criterion for each method and time step to ensure that as accurate as possible a solution was obtained. The results presented are those for which a further reduction of the convergence criterion by an order of magnitude led to a less than 1 per cent change in the solution accuracy. This degree of accuracy was obtained with different criteria for each method and each time step, ranging from 1×10^{-4} for the P1 projection method with time step $\Delta t = 0.1$ to 1×10^{-9} for the iterative method at time step $\Delta t = 0.003125$. For the iterative method each integration of the Poisson pressure correction equation was halted after five iterations of the GMRES procedure, regardless of the accuracy of the solution at that stage of integration. The divergence test is applied to the iterative method after the momentum equations have been solved, and it was required that at each time step at least two iterations of the momentum/pressure correction cycle were carried out, regardless of the divergence after the first solution of the momentum equations.

Figure 2 shows the streamfunction and temperature contours respectively for the natural convection flow at time $t = 2$. The streamfunction contours show the recirculations that are associated with each of the thermal boundary layers that form on the heated and cooled walls. The flow continues to develop from this stage with hot and cold intrusions ejected from the boundary layers travelling across the horizontal boundaries and stratifying the interior of the domain, and the two recirculations coalescing to form a single cavity scale recirculation.

Figures 3 to 5 contain the error plotted against the time step for the pressure, U-velocity and temperature respectively, for the P1 projection, Perot, P2 pressure correction, Kim Moin, explicit and iterative schemes. The pressure results include an order Δt line, while the other results include order Δt and Δt^2 lines to allow the order of accuracy of the error results to be easily estimated.

All schemes have order Δt error for the pressure. The P2 pressure correction and iterative schemes provide nearly identical results. The fully explicit and Perot methods provide equivalent results to the P2 pressure correction once the time step is small enough. The Kim and Moin scheme is apparently considerably less accurate than the iterative scheme, however the pressure that is plotted is actually ϕ. The full pressure, as given in Equation (18) in the previous section, is also plotted for comparison and denoted Kim and Moin full pressure, in the legend. This result is nearly identical to that of the P2 pressure correction result, and very close to the iterative result. The pressure plotted for the P2 pressure correction was that obtained using the simpler expression Equation (25). Results have also been obtained for the P2 pressure correction full pressure, given in Equation (24), however the results were not discernible from the simple pressure results shown and are not included. Results obtained for

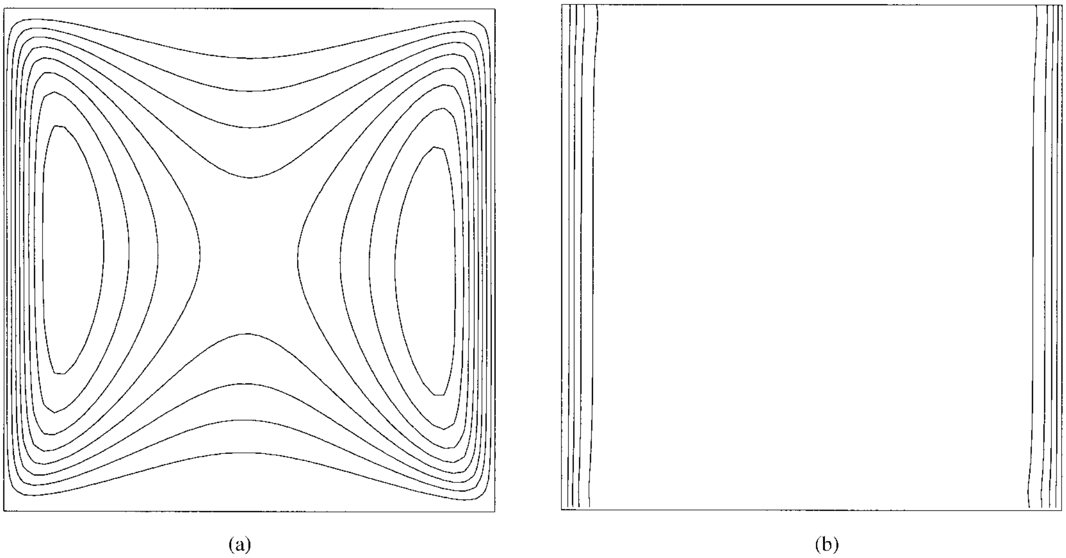

Figure 2. (a) Streamfunction contours for the natural convection flow at time $t = 2$, with 10 equally spaced contours between the maximum and minimum values of 2.3×10^{-3} and -2.0×10^{-5} respectively. (b) Temperature contours for the natural convection flow at time $t = 2$, with 10 equally spaced contours between the maximum and minimum values of 0.5 and -0.5 respectively.

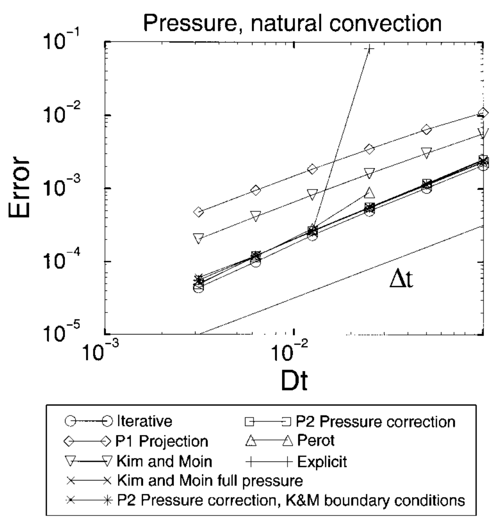

Figure 3. Pressure error variation with time step for natural convection flow.

the P2 pressure correction with Kim and Moin-type boundary conditions, as given in Equation (23), are almost identical to the standard pressure correction results. The P1 projection method is approximately an order of magnitude less accurate than the iterative method, and is the least accurate scheme. The P1 projection results shown were obtained using the simplified

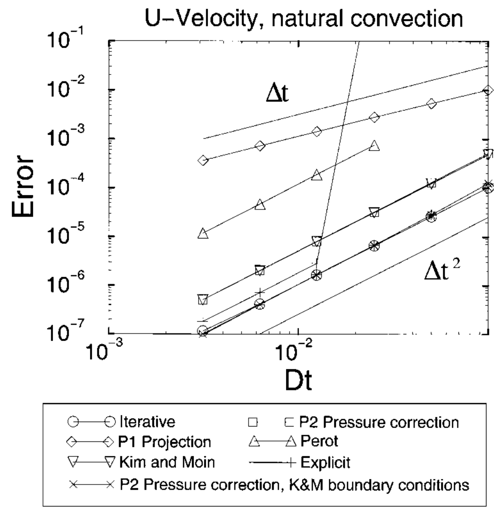

Figure 4. U-velocity error variation with time step for natural convection flow.

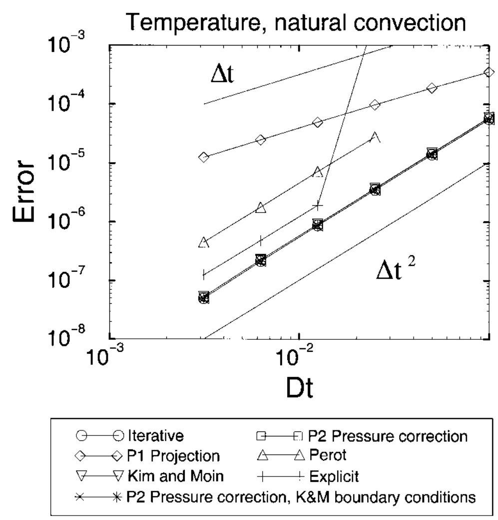

Figure 5. Temperature error variation with time step for natural convection flow.

form of pressure, Equation (15). P1 projection results have also been obtained with the full pressure, Equation (14), however these showed a proportionally small improvement and are not shown.

For the U-velocity the P1 projection method is first-order, while all other methods are providing second-order in time accuracy. The Perot method is considerably less accurate than the P1 pressure correction and Kim and Moin methods, which provide nearly identical results. The difference between the P1 pressure correction and Kim and Moin methods and the more accurate iterative method is about a factor of five times and is a result of the second-order

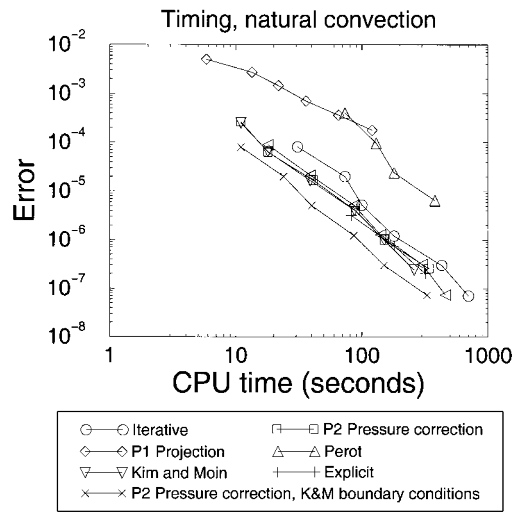

Figure 6. Timing results for natural convection flow.

error given in Equations (16) and (22). This error is converted to third-order for the P2 pressure correction method by including Kim and Moin-type boundary conditions as shown in Equation (23), and the P2 pressure correction method is then seen to give identical results to the iterative method. The explicit method provides results marginally less accurate than the iterative method once the time step is small enough. Results for the V-velocity, which for brevity are not included, show identical behaviour to those for the U-velocity.

Results for the temperature show similar behaviour to the velocity components, with the P1 projection method giving first-order accuracy and all other methods second-order. Both the P2 pressure correction methods and the Kim and Moin method are almost identical to the iterative method, indicating that the error in the velocity fields has only a small effect on the temperature. The explicit scheme is approximately a factor of two less accurate than the iterative method, while Perot's method is about an order of magnitude less accurate.

Results have also been obtained using the maximum variation between the trial and benchmark solutions as the error measure, and this maximum norm showed exactly the same behaviour as that observed for the L_2 norm. The pointwise error has also been examined at five locations. The points considered were at half the cavity height at $x = 0.04, 0.10, 0.15, 0.25, 0.35$, thereby spanning the boundary layer and cavity interior adjacent to the left hand wall. Again the pointwise error showed exactly the same behaviour as the L_2 norm, with the exception that at the interior points no temperature error was observed, as is to be expected.

Run times have been obtained for each of the methods, and are presented in Figure 6. The timings are processor times obtained running in double precision on a DEC 3000-700, and are shown as CPU time in seconds on the horizontal axis. The error for each method is the average of the velocity and temperature errors at each of the time steps for which results were obtained, and is shown on the vertical axis. Presenting the results in this form means that it is possible to compare the CPU time required for each method to achieve a given accuracy, and thus to assess the comparative efficiency of each method.

It is clear that the P1 projection and Perot methods are both relatively inefficient. The P2 pressure correction, Kim and Moin and explicit methods provide approximately the same efficiency, and are all more efficient than the iterative method. The most efficient method is the P2 pressure correction with Kim and Moin boundary conditions.

Results have also been obtained for natural convection flow with a Rayleigh number $Ra = 6 \times 10^8$. The results were obtained on a non-uniform grid of 100×100 nodes, with a mesh size of 2×10^{-3} at the walls, and an expansion ratio of 7 per cent moving away from the walls. Solutions were obtained for time steps of $\Delta t = 5 \times 10^{-3}$, $\Delta t = 2.5 \times 10^{-3}$ and $\Delta t = 1.25 \times 10^{-3}$. The solutions were compared to a benchmark solution obtained with $\Delta t = 3.125 \times 10^{-4}$ at times $t = 0.2$ and $t = 20$. The $t = 0.2$ results corresponds approximately to that shown above for the lower Rayleigh number, with the boundary layers then close to full development. By $t = 20$ the intrusions have crossed the cavity, striking the far wall, and the process of filling the cavity interior with stratified fluid has begun. For the $t = 0.2$ case the high Rayleigh number results produced identical behaviour to that observed at the lower Rayleigh number. The convergence with respect to the time step for the modified P1 projection, P2 pressure correction and iterative methods was second-order for the temperature and velocities for the L_2 norm, the maximum norm and pointwise across the boundary layer and cavity, while the pressure convergence was first-order. The P1 projection method had first-order accuracy for all fields. The $t = 20$ result also produced generally the same behaviour, with the P1 projection method first-order and all other methods second-order, however in this case the P2 pressure correction, Kim and Moin and the iterative methods produced nearly identical accuracy. The smaller variation in performance of the methods for the longer integration indicates that it is the early time solution that places the most stress on the solvers. The scaling analysis developed by Patterson and Imberger [19] showed that the smallest time scale associated with the development of this flow is that of the initial thermal boundary layer growth and thus it reasonable to expect that it is this phase of the flow that will most severely test the solvers, as has been observed.

3.2. Driven cavity

The driven cavity has frequently been used as a benchmark test for incompressible Navier–Stokes solvers [20]. Initially the fluid in the square cavity is quiescent. At time $t = 0$ the tangential velocity on the upper boundary is set to one, with the normal velocity on the upper boundary set to zero, and the other boundaries no-slip. The control parameter is the Reynolds number, which is set to $Re = 400$, based on the height of the cavity and the tangential velocity at the upper boundary. Solutions have been obtained on a 50×50 uniform grid. Convergence tests again showed that the variation between the 50×50 solution and the solution obtained on a 200×200 mesh was less than 1 per cent, and the 50×50 mesh is therefore considered to provide a sufficiently accurate resolution. The solution is integrated in time from $t = 0$ to $t = 2$, in non-dimensional units, for time steps ranging from $\Delta t = 2.5 \times 10^{-2}$ to $\Delta t = 3.125 \times 10^{-3}$. Times have been non-dimensionalized by the height of the cavity and the tangential velocity on the upper boundary. Again the error was quantified by obtaining the L_2 norm of the difference of the test solution and a benchmark solution obtained with $\Delta t = 7.8125 \times 10^{-4}$ integrated for the same amount of time. Convergence criterion used for the driven cavity were obtained in the same manner as for the natural convection flow, and range from 1.0×10^{-4} for the P1 projection method at $\Delta t = 2.5 \times 10^{-2}$ to 1.0×10^{-7} for the iterative method at

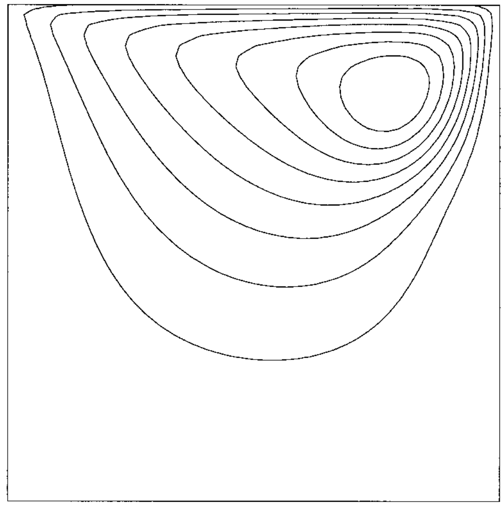

Figure 7. Streamfunction contours for the driven cavity flow at time $t=2$, with 10 equally spaced contours between the maximum and minimum values of 0.0539 and 0.0 respectively.

$\Delta t=3.125\times10^{-3}$. Again the largest time step, $\Delta t=2.5\times10^{-2}$, is near to the empirically obtained stability limit of $\Delta t=2.0\times10^{-2}$. For this flow the Perot method had approximately the same stability behaviour as the other methods while the explicit method, which was less stable, was limited to a maximum time step of $\Delta t=1.25\times10^{-2}$.

Figure 7 shows the streamfunction contours for the driven cavity at time $t=2$. Flow at the top boundary of the cavity is from left to right, and a recirculation has formed beneath the top boundary, with its centre located near to the downstream end. As the flow continues to develop the centre of the recirculation shifts towards the centre of the cavity, the flow becomes much stronger in the lower region, and small reverse recirculations develop in the bottom right and left corners.

Figures 8 and 9 contain the errors for the pressure and U-velocity component for the P1 projection, Perot, P2 pressure correction, Kim and Moin, explicit and iterative schemes. First-order accuracy for the pressure is obtained with all methods. The P1 projection method is the least accurate by approximately an order of magnitude, while the Kim and Moin method is a factor of two less accurate than the most accurate result. All other schemes provide nearly identical results. Again the apparent lower accuracy of the Kim and Moin scheme is obtained only with the incomplete pressure, when the full pressure is calculated it is almost identical to the other accurate results. Full pressure results for the P1 projection method showed a small improvement, while no discernible variation was observed in the full pressure results for the P2 pressure correction method, these results are not shown. Results obtained with the P2 pressure correction method with Kim and Moin boundary conditions were almost identical to the standard method and are not shown.

Results for the U-velocity show the P1 projection method is first-order accurate while all other methods are second-order. Perot's method is the least accurate of the second-order methods, by approximately an order of magnitude, while the explicit method is, by a small margin, the most accurate. The P2 pressure correction, Kim and Moin and iterative methods

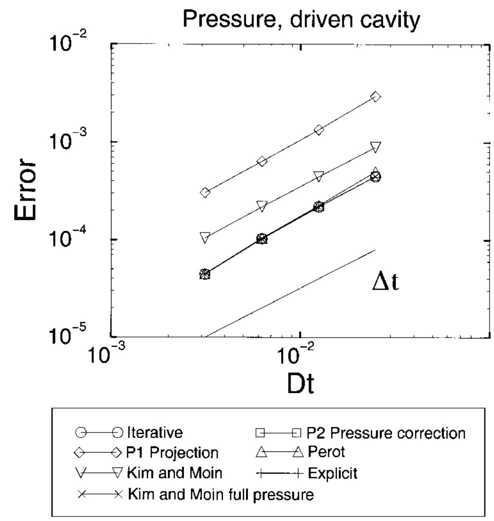

Figure 8. Pressure error variation with time step for driven cavity flow.

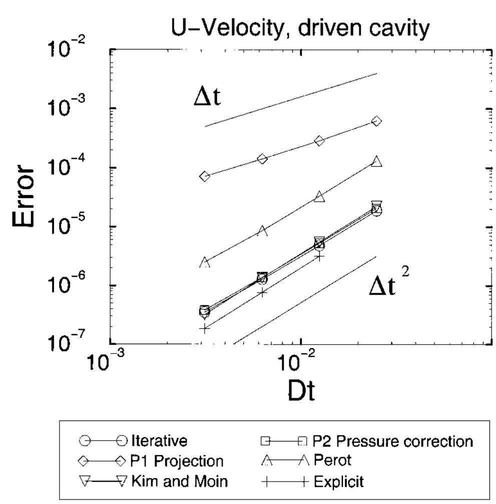

Figure 9. U-velocity error variation with time step for driven cavity flow.

are providing nearly identical results. Results obtained with the P2 pressure correction method with Kim and Moin boundary conditions were again almost identical to the standard method and are not shown. Results for the V-velocity are similar to the U-velocity and for brevity are not shown.

Results have also been obtained for the driven cavity using the maximum variation between the trial and benchmark solution as the error measure, and this maximum norm showed exactly the same behaviour as that observed for the L_2 norm. The pointwise error has also been

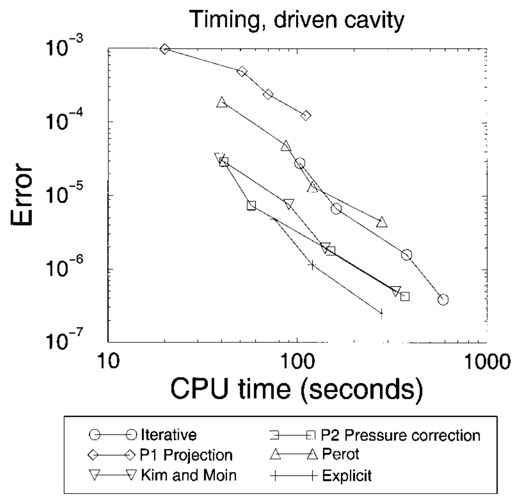

Figure 10. Timing results for driven cavity flow.

examined at the locations given above for the natural convection flow, and again showed exactly the same behaviour as the L_2 norm.

Run times versus error for each of the methods for the driven cavity are presented in Figure 10. The error is the average of the U-velocity and V-velocity errors, and the run time is in CPU seconds. The least efficient scheme is the P1 projection method, while the P2 pressure correction and Kim and Moin schemes provide a similar efficiency. For very small errors the explicit scheme is the most efficient. The Perot and iterative schemes have approximately equivalent efficiency, lying between the P1 projection and the other methods.

Results have also been obtained for driven cavity flow with a Reynolds number $Re = 5000$. The results were obtained on the same non-uniform grid as that used for the high Rayleigh number natural convection flow, and described above. Solutions were obtained for times steps of $\Delta t = 2.5 \times 10^{-3}$, $\Delta t = 1.25 \times 10^{-3}$ and $\Delta t = 6.25 \times 10^{-4}$. The solutions were compared to a benchmark solution obtained with $\Delta t = 1.5625 \times 10^{-4}$ at time $t = 0.05$. The high Reynolds number results had an identical behaviour to those obtained at the lower Reynolds number.

3.3. Zero shear boundary

The methods have been tested for a zero shear boundary by obtaining results for symmetric duct flow. In this case the domain has an aspect ratio of 10 (length/height) and the control parameter is the Reynolds number, set to $Re = 40$, based on the domain height and the inlet velocity. A fixed uniform non-dimensional horizontal velocity of 1.0 and vertical velocity of 0.0 are specified at the left hand boundary inlet, the bottom boundary has zero velocity, the upper boundary zero tangential shear and zero normal velocity and the right hand outflow boundary zero streamwise velocity derivatives. The upper boundary therefore lies on the central symmetry line for a symmetric duct flow. Initially the fluid is quiescent with the inlet boundary condition impulsively enforced at time $t = 0$ and the flow allowed to develop.

The domain has been discretized with a uniform 50×50 grid and results have been obtained for time steps in the range $\Delta t = 5.0 \times 10^{-5}$ to 1.28×10^{-2} and compared to benchmark

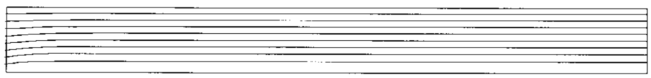

Figure 11. Streamfunction contours for the duct flow at time $t = 0.0512$, with 10 equally spaced contours between the maximum and minimum values of 0.0 and -0.098 respectively.

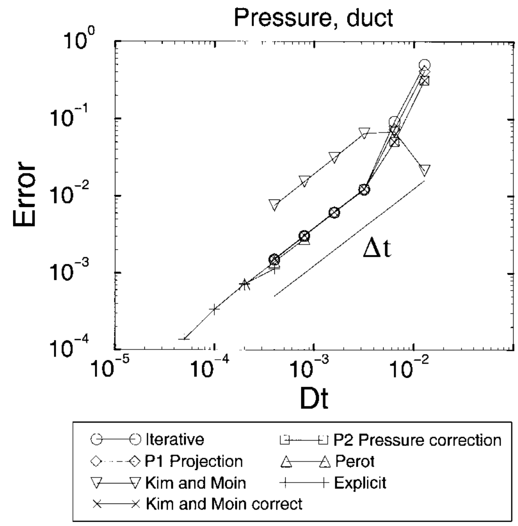

Figure 12. Pressure error variation with time step for duct flow.

results obtained for $\Delta t = 1.25 \times 10^{-5}$, all integrated from $t = 0.0$ to $t = 0.0512$. The maximum time step of 1.28×10^{-2} is close to the empirically obtained stability limit of 2.0×10^{-2}. Convergence criterion used for the duct flow were obtained in the same manner as for the natural convection flow, and range from 1.0×10^{-5} for the P1 projection method at $\Delta t = 1.28 \times 10^{-2}$ to 1.0×10^{-8} for the iterative method at $\Delta t = 5.0 \times 10^{-5}$. The stability limit for Perot's method was $\Delta t = 3.2 \times 10^{-3}$ and for the explicit method $\Delta t = 1.6 \times 10^{-3}$.

Figure 11 shows the streamfunction contours for the duct flow time $t = 0.0512$. The inlet boundary is at the left. Initially the flow is uniform and all streamfunction contours are horizontal. As the flow develops, the non-slip boundary on the bottom has led to the formation of a boundary layer region causing the streamfunction contours to curve upwards, as seen in the figure.

Figures 12 and 13 contain the errors for the pressure and U-velocity component for the P1 projection, Perot, P2 pressure correction, Kim and Moin and iterative schemes. The scheme denoted as Kim and Moin uses the incorrect tangential shear boundary condition, given in Equation (17), while the scheme using the correct tangential shear boundary condition is labelled Kim and Moin correct. For the pressure all methods are exhibiting non-asymptotic behaviour for the larger time steps, with the Kim and Moin method showing an improvement in accuracy with increasing time step. However, once the time step is small enough all schemes are first-order in time, with the Kim and Moin method the least accurate by an order of magnitude in the asymptotic region. All other methods, including the Kim and Moin

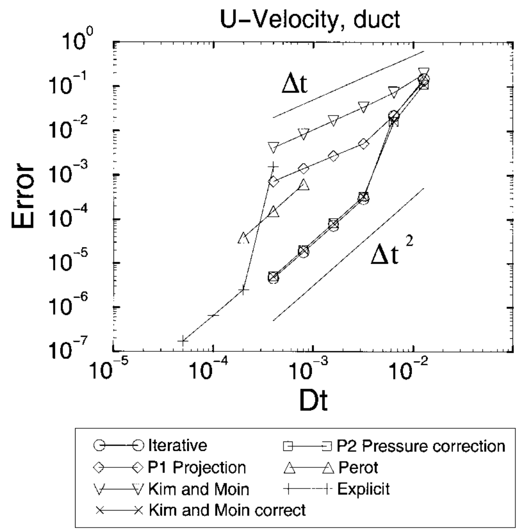

Figure 13. U-velocity error variation with time step for duct flow.

correct, provide approximately equivalent accuracy. For the duct flow no discernible variation is obtained by using the full pressure for any of the methods, or the P2 pressure correction with Kim and Moin boundary conditions, and these results are not shown.

The U-velocity results show both the P1 projection and Kim and Moin methods to be first-order once the solutions become asymptotic, with the Kim and Moin the least accurate. All other methods are second-order with the P2 pressure correction, Kim and Moin correct and iterative methods giving almost identical results. The explicit results lie on approximately the same line as the iterative results but require a very small time step for stability. The Perot method is considerably less accurate than the other second-order methods. The V-velocity results are similar to the U-velocity results and for brevity are not shown.

Results have also been obtained for the duct using the maximum variation between the trial and benchmark solutions as the error measure, and this maximum norm showed exactly the same behaviour as that observed for the L_2 norm. The pointwise error has also been examined and again showed exactly the same behaviour as the L_2 norm.

Run times versus error for each of the methods for the duct are presented in Figure 14. The error is the average of the U-velocity and V-velocity errors, and the run time is in CPU seconds. The least efficient scheme is the iterative method. It is difficult to distinguish a single most efficient scheme; however, the P2 pressure correction and Kim and Moin correct schemes are consistently efficient over a range of errors. The explicit scheme is most efficient for very small errors, while for large errors the Kim and Moin method performs well. The Perot method is very inefficient for the largest time step considered, but for the other time steps performs well. The inefficiency at the largest time step is due to the marginal stability of the Perot method at that time step, requiring many more iterations of the GMRES solver for the pressure correction equation to achieve a converged solution. This meant that the largest time step considered required a greater CPU time than the next smaller time step, leading to the dog-leg behaviour of the efficiency curve for the Perot results.

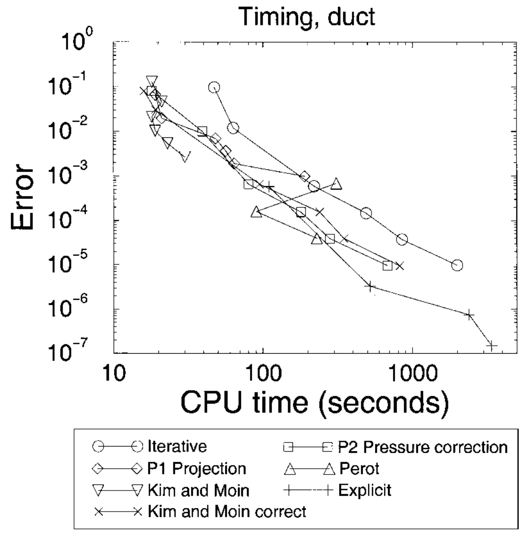

Figure 14. Timing results for duct flow.

Results have also been obtained with the Reynolds number $Re = 400$ for the same grid and time steps used for the lower Reynolds number. The high Reynolds number results produced identical behaviour to that observed at the lower Reynolds number.

4. DISCUSSION

The use of the P1 projection method with standard boundary conditions and an implicit-in-time discretization for the viscous terms means that the divergence-free velocity obtained after the projection step will not in general satisfy the momentum equations. The error obtained when the projected velocity field is substituted back into the momentum equations cannot, in discrete form, be expressed as the gradient of a scalar, and therefore cannot be absorbed into the pressure or otherwise removed. The analysis presented in Section 2 indicates that this error will be first-order in time, and that the order of this error is not related to the time accuracy of the scheme used to integrate the momentum equations. It is therefore expected that the P1 projection method with an implicit-in-time discretization for the viscous terms will, in practice, exhibit only first-order convergence with respect to the time step. This expected first-order behaviour has been confirmed by carrying out a direct accuracy analysis using the P1 projection method for natural convection, driven cavity and duct flows.

The iterative method provides an exact solution to the discretized Navier–Stokes equations, at least to the accuracy with which the system has been inverted. This scheme is therefore expected to exhibit the same time accurate behaviour as that with which the momentum equations have been solved. Results obtained with this method have shown that in practice it is second-order in time, as expected. The potential drawback of this approach is that it may require a large number of iterations at each time step, and therefore may potentially be inefficient. The iterative method represents the best solution that can be obtained with the non-iterative schemes that attempt to solve the same discrete equation set, any difference between the iterative solution and the solution obtained with the non-iterative schemes is a result of the projection error, that is the error that results when the corrected velocity field is substituted back into the discrete momentum equations.

Perot recognized this first-order projection error as a commutation error and suggested a means to remove it in the context of an LU factorization scheme, requiring no modification to the boundary conditions. Perot's suggestion has been used here with the fractional step projection method, but still required that special boundary conditions be applied to the additional term introduced into the velocity correction. The resulting modified projection method was found to be second-order in time, although for all the flows tested it is the least accurate of the second-order methods. Perot also suggested a more accurate modification to the standard projection method that reduces the projection error to third-order, however that scheme led to a very large computational molecule and complex boundary conditions when implemented with the fractional step approach, and was therefore not tested.

Kim and Moin recognized the first-order projection error as an error in the boundary conditions of the * velocity field, and derived a modified boundary condition for fixed velocity boundaries that makes the scheme overall second-order in time. In this paper it has been shown that both Perot and Kim and Moin are correct in their analysis of the projection error, it is a commutation error, as noted by Perot, and it does occur at the boundary and can be removed by modifying the * velocity boundary condition, as noted by Kim and Moin. It has also been shown in this paper that the modified Kim and Moin boundary condition need only be applied to the tangential component of the * velocity field. The Kim and Moin method does not entirely eliminate the projection error, rather the first-order error is replaced with a second-order error. This error depends on the time rate of change of spatial gradients of the pressure field, and in flows where these are significant, such as the natural convection flow, the velocities obtained with the Kim and Moin method are less accurate than the iterative scheme by up to a factor of five. In the driven cavity flow there is only a small difference between the Kim and Moin scheme and the iterative results.

The Kim and Moin results for the duct flow show that some care must be taken with the application of the modified boundary conditions. The duct flow includes a zero tangential shear boundary and Kim and Moin did not derive a modified boundary condition for this case. A naive application of the fixed velocity boundary condition results in a scheme that is first-order, whereas a proper analysis of the error shows that for this case no modification is required on the zero shear boundary. Using this approach, with Kim and Moin conditions on the other boundaries, the scheme is second-order with only a very small difference between it and the iterative results. A proper derivation of the zero shear boundary condition in the manner given in Kim and Moin also shows that no modification is required on this boundary. It is clear using this approach that appropriate modified boundary conditions must be derived for each physical velocity boundary condition.

The P2 pressure correction method has been shown to have a similar second-order projection error to that of the Kim and Moin scheme, and this is reflected in the results where there is only a very small difference in the velocity error between the two schemes, provided the correct Kim and Moin boundary condition is used for the duct flow. The advantage of the pressure correction method is that this degree of accuracy is obtained without any modification to the basic scheme. For the natural convection flow the pressure correction velocity results are up to a factor of five less accurate than the iterative results, as for the Kim and Moin results, and this variation is a result of the second-order error term given in Equation (22). This term may be modified to a third-order in time error by using the Kim and Moin boundary conditions, but with the pressure correction π, rather than with the simple pressure ϕ. This modified pressure correction method produces identical results to the iterative method for the natural convection flow. A similar modification could easily be developed for the Kim and Moin method, requiring only that $\phi^{n+1} - \phi^n$ be obtained and stored.

Results have also been obtained using a fully explicit projection method. In this scheme there is no projection error and thus the results are expected to be of comparable accuracy to those of the iterative method. This has been shown to be the case for all the flows with the explicit scheme slightly less accurate for the natural convection flow, slightly more accurate for the driven cavity, and almost identical for the duct flow. The behaviour of the explicit scheme provides further support for the hypothesis that the first-order behaviour of the basic projection method is a result of the first-order projection error.

A scheme similar to Issa's PISO scheme [14] has also been tested. This method is identical to the P2 pressure correction method, but with two iterations of the momentum/pressure correction equation system being made at each time step. Tests have shown that the two iteration scheme takes between 60 and 90 per cent more computation time per time step than the single iteration P2 pressure correction scheme. For both the driven cavity and the duct flow the P2 pressure correction method is already providing equivalent accuracy to the iterative scheme. Using two iterations of the P2 pressure correction scheme leads to no improvement in the accuracy and therefore reduces the efficiency. For the natural convection flow the iterative scheme improves the accuracy by approximately a factor of five compared to the P2 pressure correction scheme. Using two iterations of the P2 pressure correction scheme provided very close to the same accuracy as the iterative scheme. For the larger time steps the iterative scheme ran with up to 16 iterations for the smallest convergence criterion, while for the smallest time step and largest convergence criterion only two iterations per time step were carried out, which was the minimum allowed. The results obtained with a two iteration P2 pressure correction scheme show that close to optimum accuracy can be obtained for the natural convection flow with only two iterations, however the increase in computation time required for the second iteration to be carried out means that no substantial increase in efficiency is obtained even in this case.

It has also been observed in the results section that for all the methods tested, and irrespective of the accuracy of the velocity fields, and the temperature field for the natural convection flow, the pressure is always only first-order in time. Perot [7] demonstrated that for segregated Navier–Stokes solvers it is not possible to obtain better than first-order accuracy for the pressure field as a result of an order Δt difference between the source terms for the true pressure equation, obtained by taking the divergence of the momentum equations, and for the pressure or pressure correction equation, obtained by taking the divergence of the * velocity field.

The difference between the simple pressure, that is ϕ for the P1 projection and Kim and Moin methods and the pressure corrected using only π for the P2 pressure correction method, and the full pressure obtained using Equations (14) and (18) for the P1 projection and Kim and Moin methods respectively and Equation (24) for the P2 pressure correction method, has also been investigated. For the P2 pressure correction method the difference between the simple and full pressure is second-order in time, and this makes only a very small difference to the pressure field which is itself only first-order accurate. This is reflected in the very small difference between the P2 pressure correction pressure and the iterative pressure for all the flows examined. Obtaining the full pressure improved the accuracy of the Kim and Moin pressure prediction considerably, resulting in an equivalent accuracy to the P2 pressure correction method and only a very small difference with the iterative result. Using the full pressure with the P1 projection method also improves the accuracy of the pressure by a similar amount to that obtained with the Kim and Moin method, however as the P1 projection method ϕ is considerably less accurate than that of the Kim and Moin method, the same improvement in accuracy leads to a proportionally smaller change and the P1 projection full pressure is still less accurate than ϕ obtained with the Kim and Moin method.

Empirical tests have been carried out and show that all the schemes are conditionally stable, with the stability dependent on the CFL number. For all schemes the maximum CFL number attainable was approximately 1.0 when based on interior velocity values. Results for the driven cavity have been obtained with a CFL number greater than 1.0 based on the tangential top boundary velocity. However, the large vertical gradient of horizontal velocity in this region means that the horizontal velocity at the immediate interior node is considerably reduced, and the CFL number based on that velocity is less than 1.0 for the largest time step. Both the Perot and explicit methods were considerably less stable than the other schemes. For the duct flow the maximum time step attainable for Perot's scheme was $1/16$ that of the more stable schemes, while for the explicit scheme the maximum attainable timestep was $1/32$. This reduction in stability was noted by Perot, and attributed to a reduction in the positive definite nature of the scheme. The explicit method requires that an additional stability constraint be satisfied associated with the diffusion terms, and this can be considerably more restrictive than that associated with the advective terms. It is for this reason that schemes with an implicit discretization of the viscous terms are commonly used.

Overall the iterative scheme is the most accurate of those tested, in the sense that for a given time step the error is the smallest. To determine which of the schemes is the most efficient it is necessary to consider the computational effort that must be expended to obtain that accuracy. When such a comparison is made the iterative scheme is not the most efficient. For all the flows tested the P2 pressure correction and Kim and Moin schemes are more efficient than the iterative scheme, and have approximately equivalent efficiency. The explicit scheme is also efficient for small time steps but due to its relative instability it has a limited applicability. The P1 projection method and Perot's method are relatively inefficient for the natural convection and driven cavity problems, although for the duct flow they are equivalent to the P2 pressure correction and Kim and Moin methods, at least for the smaller time steps for the Perot scheme. The early time natural convection cavity flow is apparently the most complex problem in the sense that it demonstrates the greatest variation between the schemes, and for that flow the P2 pressure correction method with Kim and Moin type boundary conditions is clearly the most efficient, and since it has equivalent efficiency to the basic P2 pressure correction method for the other flows, overall this is the most efficient scheme.

5. CONCLUSIONS

The P1 projection method, in which the pressure is not included in the momentum equations and with the viscous terms differenced using an implicit in time approach, with standard boundary conditions, is first-order in time, regardless of time accuracy of the momentum solver. The first-order in time error results from the coupling of the implicit viscous term and ϕ, used to project the * velocity onto a divergence-free solution.

Perot's method includes a correction to remove the commutation error of the standard method. Implementation of this method within the fractional step context requires special attention to the boundary conditions of the additional correction term. With appropriate boundary conditions the modified scheme exhibited second-order in time accuracy, however this approach was the least accurate of the second-order schemes.

An alternative method of removing the projection error by defining modified conditions for the tangential * velocity field was suggested by Kim and Moin. This method converts the first-order projection error into an equivalent second-order term, effectively making the overall scheme second-order. The remaining error term means that for flows with complex time changing pressure field the Kim and Moin method is still considerably less accurate than the iterative scheme. Similarly the use of the simple form of pressure can lead to significant errors in the pressure field that are largely removed by using the full form of the pressure.

The use of a P2 pressure correction method, in which the previous time level pressure is included in the momentum equations, leads to a second-order in time method. The error that results from the coupling of the viscous terms and π is still present, but as π is a pressure correction term which goes like Δt, the order of the error is Δt^2. The error term is similar to that of the Kim and Moin method and the schemes have equivalent performance except with regard to the pressure accuracy. Use of the simple pressure with the pressure correction method adds a second-order error into the pressure field, and this was negligible for all the flows considered. The second-order accuracy of the pressure correction method is achieved without any special attention to the boundary conditions and this is considered to be a significant advantage of this scheme. Use of Kim and Moin-type boundary conditions with the P2 pressure correction method reduces the projection error to third-order and provides solutions of equivalent accuracy to those of the iterative method. This modified P2 pressure correction method was the most accurate of the non-iterative semi-implicit schemes.

The P2 pressure correction method with Kim and Moin boundary conditions was the most efficient of the schemes tested. The standard P2 pressure correction and Kim and Moin method have similar performance and are second in efficiency to the modified P2 pressure correction method. The P1 projection and Perot methods are overall the least efficient of those tested, while the efficiency of the iterative method lies between the P1 projection and Perot methods and the Kim and Moin and pressure correction methods.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the following support: for SA, the UPS visiting professor scheme in Civil and Environmental Engineering at Stanford University and for SA and RS, Office of Naval Research Grant N00014-99-1-0413 (Dr L. Goodman, Team Leader, Physical Oceanography) and NSF Grant ATM-0073395 (Dr R. Rogers, Program Director, Physical Meteorology).

REFERENCES

1. Harlow F, Welch E. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *Physics of Fluids* 1965; **8**:2182–2189.
2. Chorin AJ. Numerical solution of the Navier–Stokes equations. *Mathematical Computations* 1968; **22**:745–762.
3. Kim J, Moin P. Application of a fractional step method to incompressible Navier–Stokes equations. *Journal of Computational Physics* 1985; **59**:308–323.

4. Zang Y, Street R, Koseff J. A non-staggered grid, fractional step method for time-dependent incompressible Navier–Stokes equations in curvilinear coordinates. *Journal of Computational Physics* 1994; **114**:18–33.
5. Patankar SV, Spalding DB. A calculation procedure for heat mass and momentum transfer in three dimensional parabolic flow. *International Journal of Heat and Mass Transfer* 1972; **15**:1787–1805.
6. Karniadakis G, Israeli M, Orszag S. High-order splitting methods for the Navier–Stokes equations. *Journal of Computational Physics* 1991; **97**:414–443.
7. Perot JB. An analysis of the fractional step method. *Journal of Computational Physics* 1993; **108**:249–253.
8. Van Kan J. A second-order accurate pressure correction scheme for viscous incompressible flow. *SIAM Journal of Science and Statistical Computation* 1986; **7**:870–891.
9. Bell JB, Colella P. A second-order projection method for the incompressible Navier–Stokes equations. *Journal of Computational Physics* 1989; **85**:257–283.
10. Gresho P. On the theory of semi-implicit projection methods for viscous incompressible flow and its implementation via a finite element method that also introduces a nearly consistent mass matrix. Part 1: Theory. *International Journal for Numerical Methods in Fluids* 1990; **11**:587–620.
11. Guermond J-L, Quartapelle L. Calculation of incompressible viscous flows by an unconditionally stable projection FEM. *Journal of Computational Physics* 1997; **132**:12–33.
12. Tau EY. A second-order projection method for the incompressible Navier–Stokes equations in arbitrary domains. *Journal of Computational Physics* 1994; **115**:147–152.
13. Dukowicz J, Dvinsky A. Approximate factorisation as a high order splitting for the incompressible flow equations. *Journal of Computational Physics* 1992; **102**:336–347.
14. Issa RI. Solution of the implicitly discretized fluid flow equations by operator splitting. *Journal of Computational Physics* 1985; **62**:40–65.
15. Armfield SW, Street R. The fractional-step method for the Navier–Stokes equations on staggered grids: the accuracy of three variations. *Journal of Computational Physics* 1999; **153**:660–665.
16. Leonard BP. A stable and accurate convective modelling procedure based on quadratic upstream interpolation. *Computational Methods in Applied Mechanics and Engineering* 1979; **19**:59–98.
17. Patterson JC, Armfield SW. Transient features of natural convection in a cavity. *Journal of Fluid Mechanics* 1990; **219**:469:497.
18. Armfield SW, Patterson JC. Wave properties of natural convection boundary layers. *Journal of Fluid Mechanics* 1992; **239**:195–211.
19. Patterson J, Imberger J. Unsteady natural convection in a rectangular cavity. *Journal of Fluid Mechanics* 1980; **100**:65–86.
20. Perng C, Street R. The SEAFLOS numerical codes: high speed N–S solvers. *International Journal of Numerical Methods in Fluids* 1989; **9**:341–362.

